

# On the Stretch Factor of Convex Polyhedra whose Vertices are (Almost) on a Sphere

Prosenjit Bose\*   Paz Carmi†   Mirela Damian‡   Jean-Lou De Carufel§  
 Darryl Hill\*   Anil Maheshwari\*   Yuyang Liu\*   Michiel Smid\*

September 5, 2016

## Abstract

Let  $P$  be a convex simplicial polyhedron in  $\mathbb{R}^3$ . The skeleton of  $P$  is the graph whose vertices and edges are the vertices and edges of  $P$ , respectively. We prove that, if these vertices are on a sphere, the skeleton is a  $(0.999 \cdot \pi)$ -spanner. If the vertices are very close to a sphere, then the skeleton is not necessarily a spanner. For the case when the boundary of  $P$  is between two concentric spheres of radii  $r$  and  $R$ , where  $R > r > 0$ , and the angles in all faces are at least  $\theta$ , we prove that the skeleton is a  $t$ -spanner, where  $t$  depends only on  $R/r$  and  $\theta$ . One of the ingredients in the proof is a tight upper bound on the geometric dilation of a convex cycle that is contained in an annulus.

## 1 Introduction

Let  $S$  be a finite set of points in Euclidean space and let  $G$  be a graph with vertex set  $S$ . We denote the Euclidean distance between any two points  $p$  and  $q$  by  $|pq|$ . Let the length of any edge  $pq$  in  $G$  be equal to  $|pq|$ , and define the length of a path in  $G$  to be the sum of the lengths of the edges on this path. For any two vertices  $p$  and  $q$  in  $G$ , we denote by  $|pq|_G$  the length of a shortest path in  $G$  between  $p$  and  $q$ . For a real number  $t \geq 1$ , we say that  $G$  is a  $t$ -spanner of  $S$ , if  $|pq|_G \leq t|pq|$  for all vertices  $p$  and  $q$ . The *stretch factor* of  $G$  is the smallest value of  $t$  such that  $G$  is a Euclidean  $t$ -spanner of  $S$ . See [8] for an overview of results on Euclidean spanners.

---

\*School of Computer Science, Carleton University, Ottawa, Canada. These authors were supported by the Natural Sciences and Engineering Research Council of Canada. D.H. was supported by an Ontario Graduate Scholarship.

†Department of Computer Science, Ben-Gurion University of the Negev, Israel.

‡Department of Computer Science, Villanova University, Villanova, PA 19403, USA. Supported by NSF grant CCF-1218814.

§School of Electrical Engineering and Computer Science, University of Ottawa, Canada.

It is well-known that the stretch factor of the Delaunay triangulation in  $\mathbb{R}^2$  is bounded from above by a constant. The first proof of this fact is due to Dobkin *et al.* [4], who obtained an upper bound of  $(1 + \sqrt{5})\pi/2 \approx 5.08$ . The currently best known upper bound, due to Xia [11], is 1.998.

Let  $P$  be a convex simplicial polyhedron in  $\mathbb{R}^3$ , i.e., all faces of  $P$  are triangles. The *skeleton* of  $P$ , denoted by  $skel(P)$ , is the graph whose vertex and edge sets are equal to the vertex and edge sets of  $P$ .

Since there is a close connection between Delaunay triangulations in  $\mathbb{R}^2$  and convex hulls in  $\mathbb{R}^3$ , it is natural to ask if the skeleton of a convex simplicial polyhedron in  $\mathbb{R}^3$  has a bounded stretch factor. By taking a long and skinny convex polyhedron, however, this is clearly not the case.

In 1987, Raghavan suggested, in a private communication to Dobkin *et al.* [4], that the skeleton of a convex simplicial polyhedron, all of whose vertices are on a sphere, has bounded stretch factor. Consider such a polyhedron  $P$ . By a translation and scaling, we may assume that the vertex set  $S$  of  $P$  is on the unit-sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Observe that this does not change the stretch factor of  $P$ 's skeleton. It is well-known that the convex hull of  $S$  (i.e., the polyhedron  $P$ ) has the same combinatorial structure as the spherical Delaunay triangulation of  $S$ ; this was first observed by Brown [3]. Based on this, Bose *et al.* [2] showed that the proof of Dobkin *et al.* [4] can be modified to prove that the skeleton of  $P$  is a  $t$ -spanner of its vertex set  $S$ , where  $t = \frac{3\pi}{2}(1 + \pi/2) \approx 12.115$ .

In Section 2, we improve the upper bound on the stretch factor to  $0.999 \cdot \pi \approx 3.138$ . Our proof considers any two vertices  $p$  and  $q$  of  $P$  and the plane  $H_{pq}$  through  $p$ ,  $q$ , and the origin. The great arc on  $\mathbb{S}^2$  connecting  $p$  and  $q$  is contained in  $H_{pq}$ . The path on the convex polygon  $Q_{pq} = P \cap H_{pq}$  that is on the same side of  $pq$  as this great arc passes through a sequence of triangular faces of  $P$ . An edge-unfolding of these faces results in a sequence of triangles in a plane, whose circumdisks form a *chain of disks*, as defined by Xia [11]. The results of Xia then imply the upper bound of  $0.999 \cdot \pi$  on the stretch factor of the skeleton of  $P$ .

A natural question is whether a similar result holds for a convex simplicial polyhedron whose vertices are “almost” on a sphere. In Section 2.4, we show that this is not the case: We give an example of a set of points that are very close to a sphere, such that the skeleton of their convex hull has an unbounded stretch factor.

In Section 4, we consider convex simplicial polyhedra  $P$  whose boundaries are between two concentric spheres of radii  $r$  and  $R$ , where  $R > r > 0$ , that contain the common center of these spheres, and in which the angles in all faces are at least  $\theta$ . We may assume that the two spheres are centered at the origin. We present an improvement of a result by Karavelas and Guibas [7], i.e., we show that for any two vertices  $p$  and  $q$ , their shortest-path distance in the skeleton of  $P$  is at most  $(1 + 1/\sin(\theta/2))/2$  times their shortest-path distance along the surface of  $P$ . The latter shortest-path distance is at most the shortest-path distance between  $p$  and  $q$  along the boundary of the convex polygon  $Q_{pq}$  which is obtained by intersecting  $P$  with the plane through  $p$ ,  $q$ , and the origin. This convex polygon contains the origin and

its boundary is contained between the two circles of radii  $r$  and  $R$  that are centered at the origin. Grüne [6, Lemma 2.40] has shown that the stretch factor of any such polygon is at most

$$\frac{\pi R/r}{2 - (\pi/2)(R/r - 1)},$$

provided that  $R/r < 1 + 4/\pi$ . In Section 3, we improve this upper bound to

$$\sqrt{(R/r)^2 - 1} + (R/r) \arcsin(r/R),$$

which is valid, and tight, for all  $R > r > 0$ . As a result, the stretch factor of the skeleton of  $P$  is at most

$$\frac{1 + 1/\sin(\theta/2)}{2} \left( \sqrt{(R/r)^2 - 1} + (R/r) \arcsin(r/R) \right).$$

## 2 Convex Polyhedra whose Vertices are on a Sphere

In this section, we prove an upper bound on the stretch factor of the skeleton of a convex simplicial polyhedron whose vertices are on a sphere. As we will see in Section 2.2, our upper bound follows from Xia's upper bound in [11] on the stretch factor of chains of disks in  $\mathbb{R}^2$ . We start by reviewing such chains.

### 2.1 Chains of Disks

Let  $\mathcal{D} = (D_1, D_2, \dots, D_k)$  be a sequence of disks in  $\mathbb{R}^2$ , where  $k \geq 2$ . For each  $i$  with  $2 \leq i \leq k$ , define

$$C_i^{i-1} = D_{i-1} \cap \partial D_i,$$

i.e.,  $C_i^{i-1}$  is that part of the boundary of  $D_i$  that is contained in  $D_{i-1}$ . Similarly, for each  $i$  with  $1 \leq i < k$ , define

$$C_i^{i+1} = D_{i+1} \cap \partial D_i.$$

The sequence  $\mathcal{D}$  of disks is called a *chain of disks*, if

1. for each  $i$  with  $1 \leq i < k$ , the circles  $\partial D_i$  and  $\partial D_{i+1}$  intersect in exactly one or two points, and
2. for each  $i$  with  $2 \leq i < k$ , the circular arcs  $C_i^{i-1}$  and  $C_i^{i+1}$  have at most one point in common.

See Figure 1 for an example.

Let  $p$  and  $q$  be two distinct points in the plane such that

1.  $p$  is on  $\partial D_1$  and not in the interior of  $D_2$ , and
2.  $q$  is on  $\partial D_k$  and not in the interior of  $D_{k-1}$ .

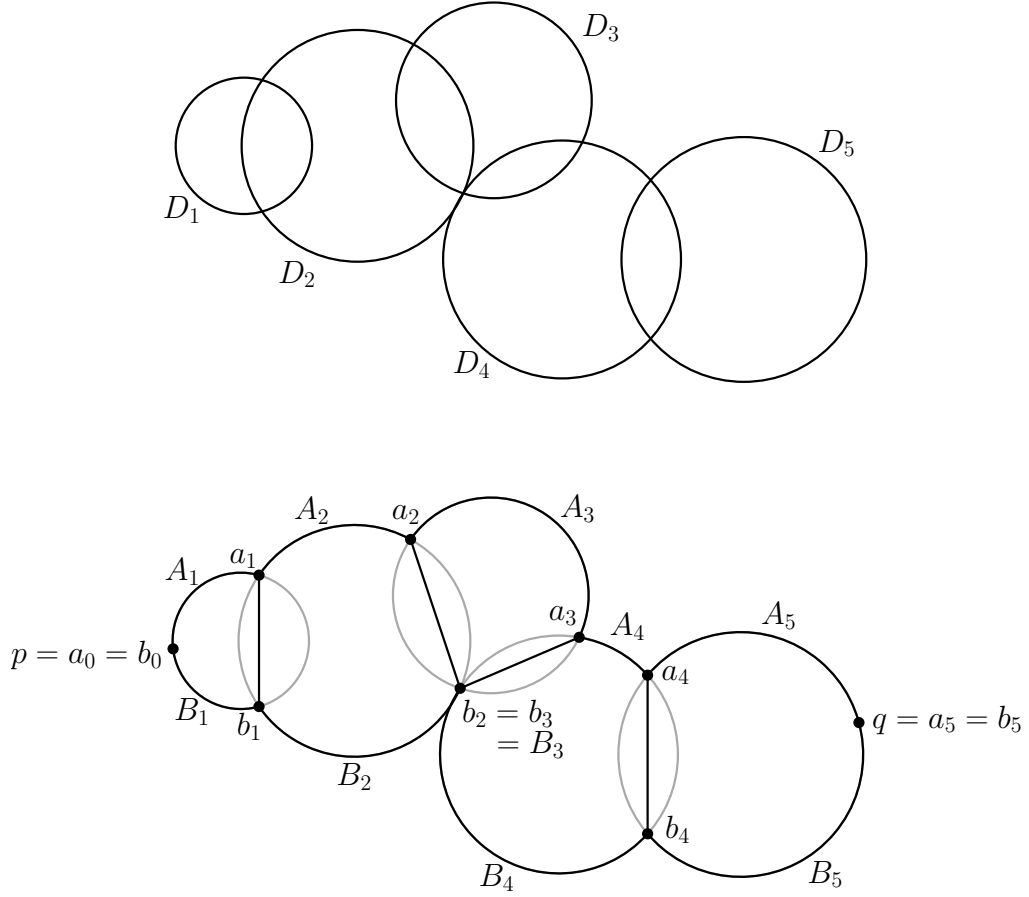


Figure 1: The top figure shows a chain  $\mathcal{D} = (D_1, D_2, \dots, D_5)$  of disks. The bottom figure shows the graph  $G(\mathcal{D}, p, q)$ ; the edges of this graph are black. The edge  $B_3$  has length zero; it consists of just the point  $b_2$  (which is equal to  $b_3$ ).

For each  $i$  with  $1 \leq i < k$ , let  $a_i$  and  $b_i$  be the intersection points of the circles  $\partial D_i$  and  $\partial D_{i+1}$ , where  $a_i = b_i$  if these two circles are tangent. We label these intersection points in such a way that  $a_i$  is on or to the left of the directed line from the center of  $D_i$  to the center of  $D_{i+1}$ , and  $b_i$  is on or to the right of this line. Define  $a_0 = p$ ,  $b_0 = p$ ,  $a_k = q$ , and  $b_k = q$ . For each  $i$  with  $1 \leq i \leq k$ , let  $A_i$  be the circular arc on  $\partial D_i$  connecting the points  $a_{i-1}$  and  $a_i$  that is completely on the same side of  $\Pi$  as  $a_{i-1}$  and  $a_i$ , and let  $B_i$  be the circular arc on  $\partial D_i$  connecting the points  $b_{i-1}$  and  $b_i$  that is on the same side of  $\Pi$  as  $b_{i-1}$  and  $b_i$ .

Consider the graph  $G(\mathcal{D}, p, q)$  with vertex set  $\{p, a_1, a_2, \dots, a_{k-1}, b_1, b_2, \dots, b_{k-1}, q\}$  and edge set consisting of

- the circular arcs  $A_1, A_2, \dots, A_k$ ,
- the circular arcs  $B_1, B_2, \dots, B_k$ , and
- the line segments  $a_1b_1, a_2b_2, \dots, a_{k-1}b_{k-1}$ .

Figure 1 shows an example.

For each  $i$  with  $1 \leq i \leq k$ , the lengths of the edges  $A_i$  and  $B_i$  are equal to the lengths  $|A_i|$  and  $|B_i|$  of these arcs, respectively. For each  $i$  with  $1 \leq i < k$ , the length of the edge  $a_i b_i$  is equal to  $|a_i b_i|$ . The length of a shortest path in  $G(\mathcal{D}, p, q)$  is denoted by  $|pq|_{G(\mathcal{D}, p, q)}$ .

**Theorem 1 (Xia [11])** *Let  $L$  be the length of any polygonal path that starts at  $p$ , ends at  $q$ , and intersects the line segments  $a_1b_1, a_2b_2, \dots, a_{k-1}b_{k-1}$  in this order. Then,*

$$|pq|_{G(\mathcal{D}, p, q)} \leq 1.998 \cdot L.$$

## 2.2 Bounding the Stretch Factor

Let  $P$  be a convex simplicial polyhedron in  $\mathbb{R}^3$  and assume that all vertices of  $P$  are on a sphere. By a translation and scaling, we may assume that this sphere is the unit-sphere  $\mathbb{S}^2$  (without changing the stretch factor of  $P$ 's skeleton). We assume that (i) no four vertices of  $P$  are co-planar and (ii) the plane through any three vertices of  $P$  does not contain the origin.

Fix two distinct vertices  $p$  and  $q$  of  $P$ . We will prove that  $|pq|_{skel(P)}$ , i.e., the length of a shortest path in the skeleton  $skel(P)$  of  $P$ , is at most  $0.999 \cdot \pi \cdot |pq|$ . If  $pq$  is an edge of  $skel(P)$ , then this claim obviously holds. We assume from now on that  $pq$  is not an edge of  $skel(P)$ .

Our proof will use the following notation (refer to Figure 2):

- $H_{pq}$ : the plane through  $p, q$ , and the origin (i.e., the center of  $\mathbb{S}^2$ ).
- $C_{pq}$ : the circle  $\mathbb{S}^2 \cap H_{pq}$ .
- $A_{pq}$ : the shorter arc of  $C_{pq}$  connecting  $p$  and  $q$ .
- $Q_{pq}$ : the convex polygon  $P \cap H_{pq}$ .

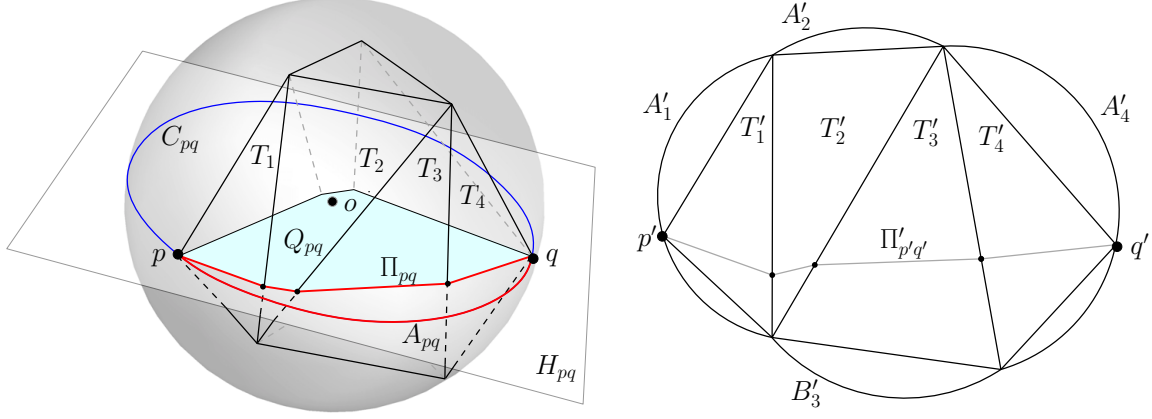


Figure 2: Illustrating the notation in Section 2.2.

- $\Pi_{pq}$ : the path along  $Q_{pq}$  from  $p$  to  $q$  that is on the same side of the line segment  $pq$  as the arc  $A_{pq}$ ; observe that  $\Pi_{pq}$  is a path between  $p$  and  $q$  along the surface of  $P$ .
- $T_1, T_2, \dots, T_k$ : the sequence of faces of  $P$  that the path  $\Pi_{pq}$  passes through. Observe that  $k \geq 2$ .

Let  $T'_1, T'_2, \dots, T'_k$  be the sequence of triangles obtained from an edge-unfolding of the triangles  $T_1, T_2, \dots, T_k$ . Thus,

- all triangles  $T'_1, T'_2, \dots, T'_k$  are contained in one plane,
- for each  $i$  with  $1 \leq i \leq k$ , the triangles  $T_i$  and  $T'_i$  are congruent, and
- for each  $i$  with  $1 \leq i < k$ , the triangles  $T'_i$  and  $T'_{i+1}$  share an edge, which is the “same” edge that is shared by  $T_i$  and  $T_{i+1}$ , and the interiors of  $T'_i$  and  $T'_{i+1}$  are disjoint.

For each  $i$  with  $1 \leq i \leq k$ , let  $D'_i$  be the circumdisk of the triangle  $T'_i$ . Let  $\mathcal{D}' = (D'_1, D'_2, \dots, D'_k)$  and let  $p'$  and  $q'$  be the vertices of  $T'_1$  and  $T'_k$  corresponding to  $p$  and  $q$ , respectively. We will prove the following lemma in Section 2.3.

**Lemma 1** *The following properties hold:*

1. *The sequence  $\mathcal{D}'$  is a chain of disks.*
2.  *$p'$  is on  $\partial D'_1$  and not in the interior of  $D'_2$ .*
3.  *$q'$  is on  $\partial D'_k$  and not in the interior of  $D'_{k-1}$ .*

Consider the graph  $G(\mathcal{D}', p', q')$  that is defined by  $\mathcal{D}'$  and the two points  $p'$  and  $q'$ ; see Section 2.1. We first observe that  $|pq|_{\text{skel}(P)}$  is at most the shortest-path distance between  $p$  and  $q$  in the graph consisting of all vertices and edges of the faces  $T_1, T_2, \dots, T_k$ . The latter shortest-path distance is equal to the shortest-path distance between  $p'$  and  $q'$  in the graph consisting of all vertices and edges of the triangles  $T'_1, T'_2, \dots, T'_k$ . Since the latter shortest-path distance is at most  $|p'q'|_{G(\mathcal{D}', p', q')}$ , it follows that

$$|pq|_{\text{skel}(P)} \leq |p'q'|_{G(\mathcal{D}', p', q')}.$$

Let  $\Pi'_{p'q'}$  be the path through  $T'_1, T'_2, \dots, T'_k$  corresponding to the path  $\Pi_{pq}$ . By Lemma 1 and Theorem 1, we have

$$|p'q'|_{G(\mathcal{D}', p', q')} \leq 1.998 \cdot |\Pi'_{p'q'}|.$$

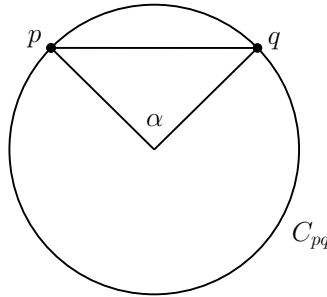
Since  $|\Pi'_{p'q'}| = |\Pi_{pq}|$ , it follows that

$$|pq|_{\text{skel}(P)} \leq 1.998 \cdot |\Pi_{pq}|.$$

It remains to bound  $|\Pi_{pq}|$  in terms of the Euclidean distance  $|pq|$ . Consider again the plane  $H_{pq}$  through  $p$ ,  $q$ , and the origin, the circle  $C_{pq} = \mathbb{S}^2 \cap H_{pq}$ , the shorter arc  $A_{pq}$  of  $C_{pq}$  connecting  $p$  and  $q$ , and the convex polygon  $Q_{pq} = P \cap H_{pq}$ . Observe that both  $p$  and  $q$  are on  $C_{pq}$ , and both these points are vertices of  $Q_{pq}$ . Moreover,  $Q_{pq}$  is contained in the disk with boundary  $C_{pq}$ . It follows that

$$|\Pi_{pq}| \leq |A_{pq}|.$$

Let  $\alpha$  be the angle between the two vectors from the origin (which is the center of  $C_{pq}$ ) to  $p$  and  $q$ ; see the figure below.



Since  $C_{pq}$  has radius 1, we have  $|A_{pq}| = \alpha$  and  $|pq| = 2 \sin(\alpha/2)$ . Therefore,

$$|A_{pq}| = \frac{\alpha/2}{\sin(\alpha/2)} \cdot |pq|.$$

The function  $g(x) = x / \sin x$  is increasing for  $0 \leq x \leq \pi/2$  (because its derivative is positive for  $0 < x \leq \pi/2$ ), implying that

$$|A_{pq}| \leq g(\pi/2) \cdot |pq| = (\pi/2) \cdot |pq|.$$

By combining the above inequalities, we obtain

$$|pq|_{skel(P)} \leq 1.998 \cdot \pi/2 \cdot |pq|.$$

Thus, assuming Lemma 1 holds, we have proved the following result.

**Theorem 2** *Let  $P$  be a convex simplicial polyhedron in  $\mathbb{R}^3$ , all of whose vertices are on a sphere. Assume that no four vertices of  $P$  are co-planar and the plane through any three vertices of  $P$  does not contain the center of the sphere. Then the skeleton of  $P$  is a  $t$ -spanner of the vertex set of  $P$ , where*

$$t = 0.999 \cdot \pi.$$

## 2.3 Proof of Lemma 1

Lemma 1 will follow from Lemma 3 below. The proof of the latter lemma uses an additional result:

**Lemma 2** *Let  $i$  be an integer with  $1 \leq i \leq k$ . The polyhedron  $P$  and the origin are in the same closed halfspace that is bounded by the plane through the face  $T_i$  of  $P$ .*

**Proof.** Let  $e_i$  be the edge of the convex polygon  $Q_{pq}$  that spans the face  $T_i$ . Since the path  $\Pi_{pq}$  (which contains  $e_i$  as an edge) is on the same side of the line segment  $pq$  as the arc  $A_{pq}$ , and since the origin is on the other side of this line segment, the polygon  $Q_{pq}$  and the origin are in the same closed halfplane (in  $H_{pq}$ ) that is bounded by the line through  $e_i$ . This implies the claim.  $\blacksquare$

**Lemma 3** *Let  $i$  be an integer with  $1 \leq i < k$  and let  $w$  be the vertex of  $T_{i+1}$  that is not a vertex of  $T_i$ . Consider the vertex  $w'$  of the unfolded triangle  $T'_{i+1}$  that corresponds to  $w$ . Then  $w'$  is not in the circumdisk  $D'_i$  of the unfolded triangle  $T'_i$ .*

**Proof.** Let  $T_i = \triangle uvq$  and  $T_{i+1} = \triangle uvw$ ; thus,  $uv$  is the edge shared by the faces  $T_i$  and  $T_{i+1}$  of  $P$ . Assume without loss of generality that  $uv$  is parallel to the  $z$ -axis. Let  $C$  be the cross-section of  $\mathbb{S}^2$  that passes through  $w$  and is orthogonal to  $uv$ . Let  $u'$ ,  $v'$ ,  $q'$ , and  $o'$  be the orthogonal projections of  $u$ ,  $v$ ,  $q$ , and  $o$  onto the plane supporting  $C$ , respectively; refer to Figure 3.

Let  $C_w$  be the circle with center  $u'$  and radius  $|u'w|$  that is coplanar with  $C$ . Since  $T'_{i+1}$  is obtained by rotating  $T_{i+1}$  about the line through  $u$  and  $v$ , it must be that  $w'$  lies on  $\partial C_w$ . Next we show that  $w'$  is exterior to  $C$ .

Let  $e$  be the intersection point between the line supporting  $o'u'$  and  $C_w$  that is farthest away from  $o'$ . By Lemma 2,  $o'$  lies interior to the convex angle  $\angle wu'q'$  and, therefore,  $e$  lies exterior to  $C$ . Moreover, the circular arc of  $\partial C_w$  with endpoints  $w$  and  $e$  that extends from  $w$  away from  $o'$  (marked as a thick curve in Figure 3b), lies exterior of  $C$ . This circular arc is precisely the locus of  $w'$ . It follows that  $w'$  is exterior to  $C$ , which implies that  $w'$  is exterior



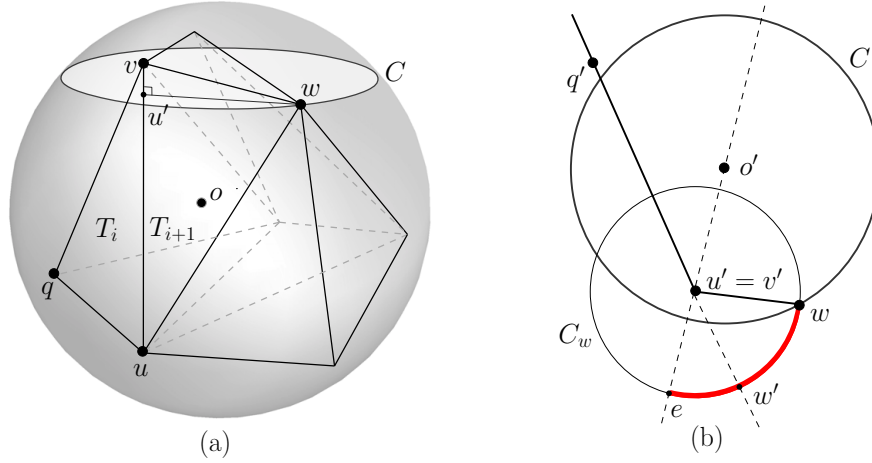


Figure 3: Cross-section through  $w$  orthogonal to  $uv$  (a) side view of  $C$  (b) top view of  $C$ .

to  $\mathbb{S}^2$ . Now observe that  $D'_i$  is congruent with the disk  $D_i$  obtained by intersecting  $\mathbb{S}^2$  with the plane supporting  $T_i$ . Since  $w'$  is coplanar with  $D_i$  and exterior to  $D_i$ , we have that  $w'$  is exterior to  $D'_i$ . This concludes the proof.  $\blacksquare$

It is easy to see that Lemma 3 implies that the sequence  $\mathcal{D}'$  is a chain of disks, i.e., this sequence satisfies the two properties given in Section 2.1. Moreover, it follows from Lemma 3 that  $p'$  is on  $\partial D'_1$  and not in the interior of  $D'_2$  and  $q'$  is on  $\partial D'_k$  and not in the interior of  $D'_{k-1}$ . Thus, we have completed the proof of Lemma 1.

## 2.4 Convex Polyhedra whose Vertices are Almost on a Sphere

In this section, we give an example of a convex simplicial polyhedron whose vertices are “almost” on a sphere and whose skeleton has unbounded stretch factor. For simplicity of notation, we consider the sphere

$$\mathbb{S}_3^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 3\}.$$

Let  $k$  be a large integer and let  $S_k$  be the subset of  $\mathbb{R}^3$  consisting of the following 12 points:

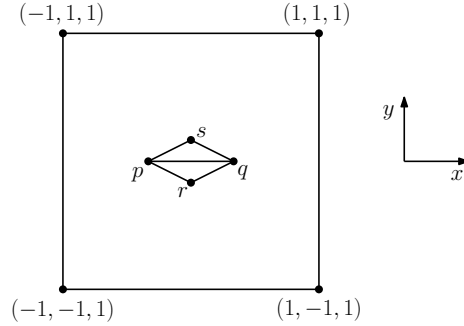
- The 8 vertices of the cube  $[-1, 1]^3$ ,
- $p = (-1/k, 0, a)$ , where  $a = \sqrt{3 - 1/k^2}$ ,
- $q = (1/k, 0, a)$ ,
- $r = (0, -b, c)$ , where  $b = 1/k^2$  and  $c = a - 1/k^3$ , and
- $s = (0, b, c)$ .

The 8 vertices of the cube and the points  $p$  and  $q$  are on the sphere  $\mathbb{S}_3^2$ . Since

$$\begin{aligned} b^2 + c^2 &= 1/k^4 + a^2 - 2a/k^3 + 1/k^6 \\ &< 1/k^4 + a^2 + 1/k^6 \\ &< a^2 + 2/k^6 \\ &= 3 - 1/k^2 + 2/k^4 \\ &< 3, \end{aligned}$$

the points  $r$  and  $s$  are in the interior of, but very close to, this sphere.

Let  $P_k$  be the convex hull of the point set  $S_k$ . Below, we will show that, for sufficiently large values of  $k$ , (i)  $(p, q, r)$  and  $(p, q, s)$  are faces of the polyhedron  $P_k$  and (ii)  $rs$  is not an edge of  $P_k$ . Thus, the figure below shows (part of) the top view (in the negative  $z$ -direction) of  $P_k$ .



The shortest path between  $r$  and  $s$  in the skeleton of  $P_k$  has length

$$|rp| + |ps| = 2|rp|,$$

which is at least twice the distance between  $r$  and  $p$  in the  $x$ -direction, which is  $2/k$ . Since  $|rs| = 2b = 2/k^2$ , it follows that the stretch factor of the skeleton of  $P_k$  is at least

$$\frac{2/k}{2/k^2} = k.$$

Thus, by letting  $k$  go to infinity, the stretch factor of  $skel(P_k)$  is unbounded.

It remains to prove that  $(p, q, r)$  and  $(p, q, s)$  are faces of  $P_k$  and  $rs$  is not an edge of  $P_k$ . The plane through  $p$ ,  $q$ , and  $s$  has equation

$$z = a - \frac{a - c}{b} \cdot y.$$

To prove that  $(p, q, s)$  is a face of  $P_k$ , it suffices to show that the points  $r$  and  $(1, 1, 1)$  are below this plane. The point  $r$  is below this plane if and only if

$$a - \frac{a - c}{b} \cdot (-b) > c,$$

which is equivalent to  $a > c$ , which obviously holds. The point  $(1, 1, 1)$  is below this plane if and only if

$$a - \frac{a - c}{b} > 1,$$

which is equivalent to

$$ab - a + c > b. \tag{1}$$

Using the fact that, for sufficiently large values of  $k$ ,  $a > 3/2$ , we have

$$\begin{aligned} ab - a + c &= a/k^2 - 1/k^3 \\ &> 3/(2k^2) - 1/k^3 \\ &> 1/k^2 \\ &= b, \end{aligned}$$

proving the inequality in (1). Thus,  $(p, q, s)$  is a face of  $P_k$ . By a symmetric argument,  $(p, q, r)$  is a face of  $P_k$  as well.

We finally show that  $(r, s)$  is not an edge of  $P_k$ . For sufficiently large values of  $k$ , both points  $r$  and  $s$  are above (with respect to the  $z$ -direction) the points  $(-1, 1, 1)$  and  $(1, 1, 1)$ . Observe that both  $r$  and  $s$  are below  $p$  and  $q$ . It follows that for any plane through  $r$  and  $s$ , (i)  $(-1, 1, 1)$  and  $q$  are on opposite sides or (ii)  $(1, 1, 1)$  and  $p$  are on opposite sides. Therefore,  $(r, s)$  is not an edge of  $P_k$ .

Let  $\alpha_k$  be the smallest angle in the face  $(p, q, s)$  of the polyhedron  $P_k$ , i.e.,  $\alpha_k = \angle(qps)$ . Then

$$(q - p) \cdot (s - p) = |pq| |ps| \cos \alpha_k,$$

where  $\cdot$  denotes the dot-product. A straightforward calculation shows that

$$\cos \alpha_k = 1 - \Theta(1/k^2),$$

implying that  $\alpha_k$  is proportional to  $1/k$ . Thus, as  $k$  tends to infinity, the smallest angle in any face of  $P_k$  tends to zero.

Observe that the polyhedron  $P_k$  does not satisfy the assumptions in Theorem 2. By a sufficiently small perturbation of the points of  $S_k$ , however, we obtain a polyhedron that does satisfy these assumptions and whose skeleton has unbounded stretch factor. We conclude that Theorem 2 does not hold for all convex simplicial polyhedra whose vertices are very close to a sphere.

### 3 Convex Cycles in an Annulus

Let  $r$  and  $R$  be real numbers with  $R > r > 0$ . Define  $Ann_{r,R}$  to be the *annulus* consisting of all points in  $\mathbb{R}^2$  that are on or between the two circles of radii  $r$  and  $R$  that are centered at the origin. Thus,

$$Ann_{r,R} = \{(x, y) \in \mathbb{R}^2 : r \leq x^2 + y^2 \leq R^2\}.$$

We will refer to the circles of radii  $r$  and  $R$  that are centered at the origin as the *inner circle* and the *outer circle* of the annulus, respectively.

In this section, we consider convex polygons  $Q$  that contain the origin and whose boundary is in  $Ann_{r,R}$ . The skeleton  $skel(Q)$  of such a polygon is the graph whose vertex and edge sets are the vertex and edge sets of  $Q$ , respectively.

Throughout this section, we will use the function  $f$  defined by

$$f(x) = \sqrt{x^2 - 1} + x \cdot \arcsin(1/x)$$

for  $x \geq 1$ .

**Lemma 4** *The function  $f$  is increasing for  $x \geq 1$ .*

**Proof.** The derivative of  $f$  is given by

$$f'(x) = \frac{x-1}{\sqrt{x^2-1}} + \arcsin(1/x).$$

It is clear that  $f'(x) > 0$  for  $x > 1$ . ■

We will prove the following result:

**Theorem 3** *Let  $r$  and  $R$  be real numbers with  $R > r > 0$  and let  $Q$  be a convex polygon that contains the origin in its interior and whose boundary is contained in the annulus  $Ann_{r,R}$ . Then the skeleton of  $Q$  is an  $f(R/r)$ -spanner of the vertex set of  $Q$ .*

Theorem 3 refers to the stretch factor of  $skel(Q)$ , which is the maximum value of  $|pq|_{skel(Q)}/|pq|$  over all pairs of distinct vertices  $p$  and  $q$  of  $Q$ . It turns out that the proof becomes simpler if we also consider points that are in the interior of edges. This gives rise to the notion of geometric dilation, which we recall in the following subsection.

### 3.1 Geometric Dilation of Convex Cycles

Let  $C$  be a convex cycle in  $\mathbb{R}^2$ . We observe that  $C$  is rectifiable, i.e., its length, denoted by  $|C|$ , is well-defined; see, for example, Section 1.5 in Toponogov [10]. For any two distinct points  $p$  and  $q$  on  $C$ , there are two paths along  $C$  that connect  $p$  and  $q$ . We denote the length of the shorter of these two paths by  $|pq|_C$ . The *geometric dilation* of  $C$  is defined as

$$Dil(C) = \max_{p,q \in C, p \neq q} \frac{|pq|_C}{|pq|}.$$

Ebbers-Baumann *et al.* [5] have proved that, for a convex cycle  $C$ ,  $Dil(C)$  is well-defined. That is, the maximum in the definition of  $Dil(C)$  exists.

Let  $p$  and  $q$  be two points on  $C$ . We say that these two points form a *halving pair* if the two paths along  $C$  between  $p$  and  $q$  have the same length.

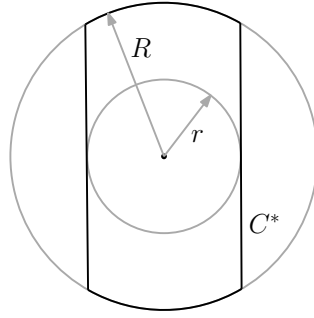
**Lemma 5 (Ebbers-Baumann *et al.* [5])** *Let  $C$  be a convex cycle in  $\mathbb{R}^2$ , and let  $h$  be the minimum Euclidean distance between the points of any halving pair. Then the geometric dilation of  $C$  is attained by a halving pair with Euclidean distance  $h$  and*

$$Dil(C) = \frac{|C|/2}{h}.$$

### 3.2 Convex Cycles in an Annulus

In this section, we consider convex cycles  $C$  that contain the origin in their interior and that are contained in the annulus  $Ann_{r,R}$ . We will prove that the geometric dilation of such a cycle is at most  $f(R/r)$ , where  $f$  is the function defined in the beginning of Section 3. Clearly, this result will imply Theorem 3.

We start by giving an example of a convex cycle whose geometric dilation is equal to  $f(R/r)$ . Let  $C^*$  be the convex cycle that consists of the two vertical tangents at the inner circle of  $Ann_{r,R}$  that have their endpoints at the outer circle, and the two arcs on the outer circle that connect these tangents; see the figure below.



A simple calculation shows that the length of  $C^*$  satisfies

$$|C^*| = 4\sqrt{R^2 - r^2} + 4R \cdot \arcsin(r/R) = 4r \cdot f(R/r).$$

**Lemma 6** *The geometric dilation of  $C^*$  satisfies*

$$Dil(C^*) = f(R/r).$$

**Proof.** Consider any halving pair  $p, q$  of  $C^*$ . Since  $C^*$  is centrally symmetric with respect to the origin, we have  $q = -p$ . The inner circle of  $Ann_{r,R}$  is between the two lines through  $p$  and  $q$  that are orthogonal to the line segment  $pq$ . Therefore,  $|pq| \geq 2r$ . Thus, by Lemma 5,

$$Dil(C^*) \leq \frac{|C^*|}{4r} = f(R/r).$$

If we take for  $p$  and  $q$  the leftmost and rightmost points of the inner circle, then  $|pq|_{C^*}/|pq| = f(R/r)$ . Therefore, we have  $Dil(C^*) = f(R/r)$ . ■

In the following lemmas, we consider special types of convex cycles in  $Ann_{r,R}$ . For each such type, we prove an upper bound of  $f(R/r)$  on their geometric dilation. In Theorem 4, we will consider the general case and reduce the problem of bounding the geometric dilation to one of the special types.

**Lemma 7** *Let  $C$  be a convex cycle in  $Ann_{r,R}$  that contains the origin in its interior, and let  $p$  and  $q$  be two distinct points on  $C$  such that  $Dil(C) = |pq|_C/|pq|$ . If both  $p$  and  $q$  are on the outer circle of  $Ann_{r,R}$ , then  $Dil(C) \leq f(R/r)$ .*

**Proof.** Let  $C'$  denote the outer circle of  $Ann_{r,R}$ . Then

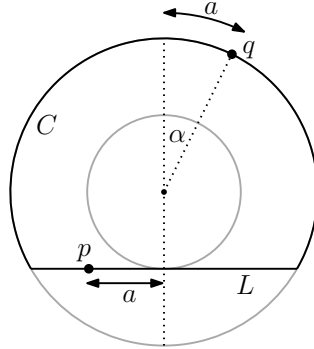
$$Dil(C) = \frac{|pq|_C}{|pq|} \leq \frac{|pq|_{C'}}{|pq|} \leq Dil(C') = \pi/2 = f(1).$$

Since, by Lemma 4,  $f(1) \leq f(R/r)$ , it follows that  $Dil(C) \leq f(R/r)$ . ■

**Lemma 8** *Consider a line segment  $L$  that is tangent to the inner circle of  $Ann_{r,R}$  and has both endpoints on the outer circle. Let  $C$  be the convex cycle that consists of  $L$  and the longer arc on the outer circle that connects the endpoints of  $L$ . Then*

$$Dil(C) \leq f(R/r).$$

**Proof.** We may assume without loss of generality that  $L$  is horizontal and touches the lowest point of the inner circle; see the figure below.



Let  $p$  and  $q$  form a halving pair of  $C$  that attains the geometric dilation of  $C$ . Observe that at least one of  $p$  and  $q$  is on the outer circle of  $Ann_{r,R}$ . If both  $p$  and  $q$  are on the outer circle, then  $Dil(C) \leq f(R/r)$  by Lemma 7. Otherwise, we may assume without loss of generality that (i)  $p$  is on  $L$  and on or to the left of the  $y$ -axis, and (ii)  $q$  is on the outer circle, on or to the right of the  $y$ -axis and above the  $x$ -axis.

We first prove that  $|pq| \geq R + r$ . Let  $p$  have coordinates  $p = (-a, -r)$  for some real number  $a$  with  $0 \leq a \leq \sqrt{R^2 - r^2}$ , and let  $\alpha$  be the angle between the  $y$ -axis and the vector from the origin to  $q$ ; see the figure above. Since  $p$  and  $q$  form a halving pair, the clockwise arc

from the highest point on the outer circle to the point  $q$  has length  $a$ . Therefore,  $\alpha = a/R$  and, thus, the coordinates of the point  $q$  are  $q = (R \cdot \sin(a/R), R \cdot \cos(a/R))$ . If we define the function  $g$  by

$$g(a) = (a + R \cdot \sin(a/R))^2 + (r + R \cdot \cos(a/R))^2$$

for  $0 \leq a \leq \sqrt{R^2 - r^2}$ , then  $|pq|^2 = g(a)$ . The derivative of  $g$  satisfies

$$g'(a) = 2a + 2(R - r) \cdot \sin(a/R) + 2a \cdot \cos(a/R),$$

which is positive for  $0 < a \leq \sqrt{R^2 - r^2}$ . Therefore, the function  $g$  is increasing and

$$|pq|^2 = g(a) \geq g(0) = (R + r)^2,$$

implying that  $|pq| \geq R + r$ .

We conclude that

$$Dil(C) = \frac{|pq|_C}{|pq|} = \frac{|C|/2}{|pq|} \leq \frac{|C|}{2(R + r)}.$$

To complete the proof, it suffices to show that

$$\frac{|C|}{2(R + r)} \leq f(R/r). \quad (2)$$

We observe that the length of  $C$  satisfies

$$|C| = 2\sqrt{R^2 - r^2} + 2R \cdot \arcsin(r/R) + \pi R.$$

Recall that

$$f(x) = \sqrt{x^2 - 1} + x \cdot \arcsin(1/x).$$

Thus, (2) becomes

$$\frac{\sqrt{R^2 - r^2} + R \cdot \arcsin(r/R) + \pi R/2}{R + r} \leq \frac{\sqrt{R^2 - r^2} + R \cdot \arcsin(r/R)}{r}.$$

The latter inequality is equivalent to

$$\pi/2 \leq \frac{\sqrt{R^2 - r^2} + R \cdot \arcsin(r/R)}{r},$$

i.e.,

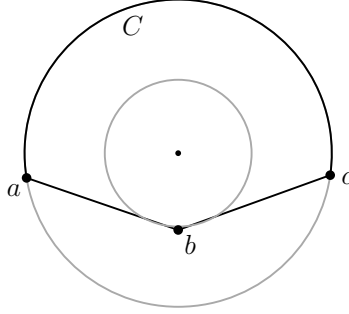
$$f(1) \leq f(R/r).$$

Since the latter inequality follows from Lemma 4, we have shown that (2) holds. ■

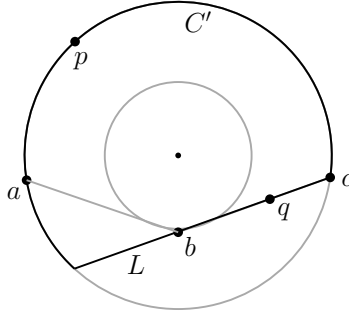
**Lemma 9** *Let  $b$  be a point in  $Ann_{r,R}$  that is on the negative  $y$ -axis. Let  $a$  and  $c$  be two points on the outer circle of  $Ann_{r,R}$  such that (i) both  $a$  and  $c$  have the same  $y$ -coordinate and are below the  $x$ -axis and (ii) both line segments  $ab$  and  $bc$  are tangent to the inner circle of  $Ann_{r,R}$ . Let  $C$  be the convex cycle that consists of the two line segments  $ab$  and  $bc$ , and the longer arc on the outer circle that connects  $a$  and  $c$ . Then*

$$Dil(C) \leq f(R/r).$$

**Proof.** The figure below illustrates the situation.



Let  $p$  and  $q$  form a halving pair of  $C$  that attains the geometric dilation of  $C$ . Observe that at least one of  $p$  and  $q$  is on the outer circle of  $Ann_{r,R}$ . If both  $p$  and  $q$  are on the outer circle of  $Ann_{r,R}$ , then  $Dil(C) \leq f(R/r)$  by Lemma 7. Otherwise, we may assume without loss of generality that  $p$  is on the outer circle of  $Ann_{r,R}$  and  $q$  is on the line segment  $bc$ , as in the figure below.



Let  $L$  be the maximal line segment in  $Ann_{r,R}$  that contains the segment  $bc$ . Let  $C'$  be the convex cycle consisting of  $L$  and the longer arc on the outer circle connecting the two endpoints of  $L$ . Then

$$Dil(C) = \frac{|pq|_C}{|pq|} \leq \frac{|pq|_{C'}}{|pq|} \leq Dil(C').$$

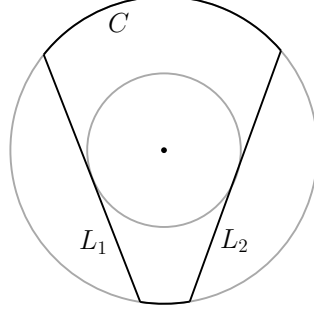
By Lemma 8, we have  $Dil(C') \leq f(R/r)$ . It follows that  $Dil(C) \leq f(R/r)$ . ■



**Lemma 10** *Consider two non-crossing line segments  $L_1$  and  $L_2$  that are tangent to the inner circle of  $\text{Ann}_{r,R}$  and have their endpoints on the outer circle. Let  $C$  be the convex cycle that consists of  $L_1$ ,  $L_2$ , and the two arcs on the outer circle that connect  $L_1$  and  $L_2$ ; one of these two arcs may consist of a single point. Then*

$$\text{Dil}(C) \leq f(R/r).$$

**Proof.** We may assume without loss of generality that  $C$  is symmetric with respect to the  $y$ -axis and  $L_1$  is to the left of  $L_2$ , as in the figure below.



If  $L_1$  and  $L_2$  are parallel, then the claim follows from Lemma 6. Thus, we assume that  $L_1$  and  $L_2$  are not parallel. We may assume without loss of generality that the length of the lower arc of  $C$  is less than the length of the upper arc, as in the figure above. We observe that

$$|C| = 4r \cdot f(R/r),$$

i.e., the length of  $C$  is equal to the length of the cycle  $C^*$  in Lemma 6. Indeed, if we rotate  $L_2$ , while keeping it tangent to the inner circle, until it becomes parallel to  $L_1$ , then the length of the cycle does not change.

Let  $p$  and  $q$  form a halving pair of  $C$  that attains the geometric dilation of  $C$ , i.e.,

$$\text{Dil}(C) = \frac{|pq|_C}{|pq|} = \frac{|C|/2}{|pq|}.$$

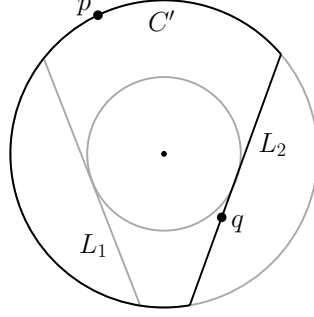
We consider three cases for the locations of  $p$  and  $q$  on  $C$ .

**Case 1:** Both  $p$  and  $q$  are on the outer circle of  $\text{Ann}_{r,R}$ .

Then we have  $\text{Dil}(C) \leq f(R/r)$  by Lemma 7.

**Case 2:**  $p$  is on the outer circle of  $\text{Ann}_{r,R}$  and  $q$  is not on the outer circle.

Since  $p$  and  $q$  form a halving pair,  $p$  must be on the upper arc of  $C$ . We may assume without loss of generality that  $q$  is on  $L_2$ . Let  $C'$  be the convex cycle consisting of  $L_2$  and the longer arc on the outer circle connecting the two endpoints of  $L_2$ ; see the figure below.



We have

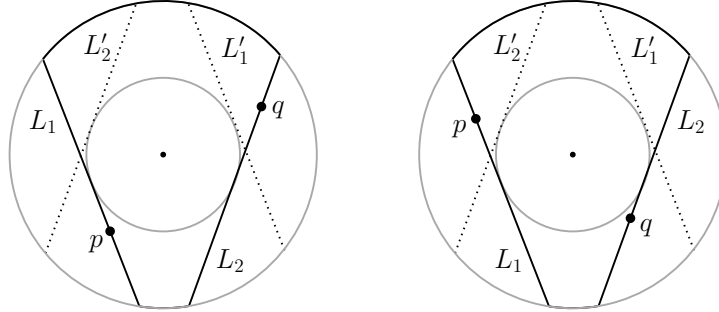
$$Dil(C) = \frac{|pq|_C}{|pq|} \leq \frac{|pq|_{C'}}{|pq|} \leq Dil(C').$$

By Lemma 8, we have  $Dil(C') \leq f(R/r)$ . It follows that  $Dil(C) \leq f(R/r)$ .

**Case 3:** Neither  $p$  nor  $q$  is on the outer circle of  $Ann_{r,R}$ .

Since  $p$  and  $q$  form a halving pair, these two points cannot both be on the same line segment of  $C$ . We may assume without loss of generality that  $p$  is on  $L_1$  and  $q$  is on  $L_2$ .

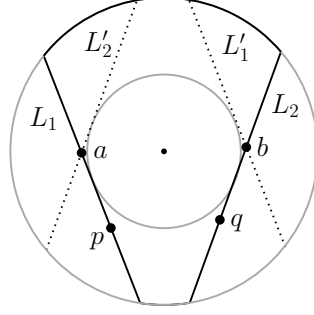
Let  $L'_1$  be the maximal line segment in  $Ann_{r,R}$  that is parallel and not equal to  $L_1$  and that touches the inner circle. Let  $L'_2$  be the maximal line segment in  $Ann_{r,R}$  that is parallel and not equal to  $L_2$  and that touches the inner circle.



We claim that  $q$  is to the right of  $L'_1$  or  $p$  is to the left of  $L'_2$ ; see the two figures above. Assuming this is true, it follows that  $|pq| \geq 2r$  and

$$Dil(C) = \frac{|C|/2}{|pq|} \leq \frac{|C|}{4r} = f(R/r).$$

To prove the claim, assume that  $q$  is to the left of  $L'_1$  and  $p$  is to the right of  $L'_2$ . Let  $a$  be the intersection between  $L_1$  and  $L'_2$ , and let  $b$  be the intersection between  $L'_1$  and  $L_2$ ; see the figure below.



Observe that both  $a$  and  $b$  are on the  $x$ -axis, and both  $p$  and  $q$  are below the  $x$ -axis. Therefore, the part of  $C$  below the  $x$ -axis is shorter than the part of  $C$  above the  $x$ -axis. Thus, the two paths along  $C$  between  $p$  and  $q$  do not have the same lengths. This contradicts our assumption that  $p$  and  $q$  form a halving pair of  $C$ . ■

We are now ready to consider an arbitrary convex cycle  $C$  that contains the origin in its interior and that is contained in  $Ann_{r,R}$ . A *homothet* of  $C$  is obtained by scaling  $C$  with respect to the origin, followed by a translation. Observe that the dilation of a homothet of  $C$  is equal to the dilation of  $C$ .

**Theorem 4** *Let  $r$  and  $R$  be real numbers with  $R > r > 0$  and let  $C$  be a convex cycle that contains the origin in its interior and that is contained in the annulus  $Ann_{r,R}$ . Then*

$$Dil(C) \leq f(R/r),$$

where  $f$  is the function defined in the beginning of Section 3.

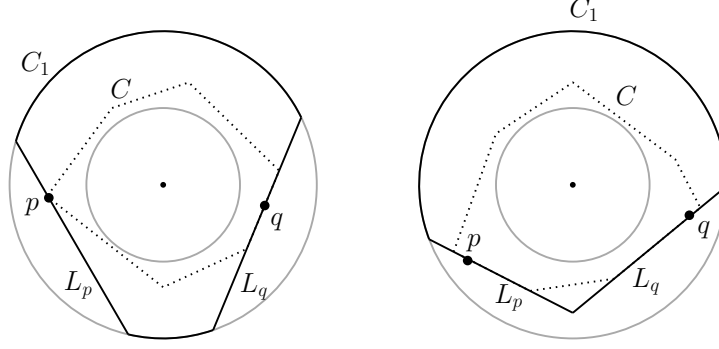
**Proof.** Let  $p$  and  $q$  form a halving pair of  $C$  that attains the geometric dilation of  $C$ , i.e.,

$$Dil(C) = \frac{|pq|_C}{|pq|}.$$

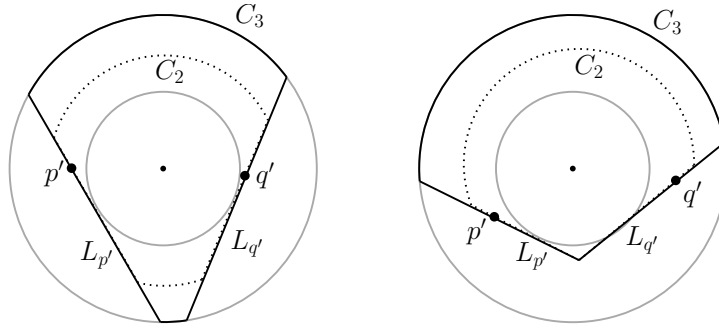
We first assume that neither  $p$  nor  $q$  is on the outer circle of  $Ann_{r,R}$ .

Let  $L_p$  and  $L_q$  be supporting lines of  $C$  through  $p$  and  $q$ , respectively. Since  $p$  and  $q$  form a halving pair,  $L_p \neq L_q$ . Let  $C_1$  be the convex cycle of maximum length in  $Ann_{r,R}$  that is between  $L_p$  and  $L_q$ . Observe that  $C_1$  contains two line segments such that (i) all their four endpoints are on the outer circle (as in the left figure below) or (ii) two of their endpoints are on the outer circle, whereas the other two endpoints meet in the interior of  $Ann_{r,R}$  (as in the right figure below). If (i) holds, we say that  $C_1$  is of *type 1*. In the other case, i.e., if (ii) holds, we say that  $C_1$  is of *type 2*. We have

$$Dil(C) = \frac{|pq|_C}{|pq|} \leq \frac{|pq|_{C_1}}{|pq|}.$$



We claim that there is a homothet  $C_2$  of  $C_1$  that is contained in  $Ann_{r,R}$  and that touches the inner circle in two points; see the two figures below.



To obtain such a homothet  $C_2$ , we do the following. First, we shrink  $C_1$ , i.e., scale it (with respect to the origin) by a factor of less than one, until it touches the inner circle. At this moment, one of the lines  $L_p$  and  $L_q$  in the shrunk copy of  $C_1$  touches the inner circle. Assume, without loss of generality, that  $L_q$  touches the inner circle, whereas  $L_p$  does not. Let  $c$  denote the “center” of the scaled copy of  $C_1$ , which is the origin. We translate  $c$  towards  $L_q$  in the direction that is orthogonal to  $L_q$ . During this translation, we shrink  $C_1$  (with respect to its center  $c$ ) while keeping  $L_q$  on its boundary. We stop translating  $c$  as soon as  $L_p$  touches the inner circle of  $Ann_{r,R}$ . The resulting translated and shrunk copy of  $C_1$  is the homothet  $C_2$ .

Let  $p'$  and  $q'$  be the two points on the homothet  $C_2$  that correspond to  $p$  and  $q$ , respectively. Then

$$Dil(C) \leq \frac{|pq|_{C_1}}{|pq|} = \frac{|p'q'|_{C_2}}{|p'q'|}.$$

Let  $L_{p'}$  and  $L_{q'}$  be supporting lines of  $C_2$  through  $p'$  and  $q'$ , respectively, and let  $C_3$  be the convex cycle of maximum length in  $Ann_{r,R}$  that is between  $L_{p'}$  and  $L_{q'}$ ; see the two figures above. Observe that  $C_3$  is either of type 1 or of type 2. In fact,  $C_3$  may be of type 2, even if  $C_1$  is of type 1. We have

$$Dil(C) \leq \frac{|p'q'|_{C_2}}{|p'q'|} \leq \frac{|p'q'|_{C_3}}{|p'q'|}.$$

First assume that  $C_3$  is of type 1. Thus, all four endpoints of the two line segments of  $C_3$  are on the outer circle of  $Ann_{r,R}$  (as in the left figure above). Then  $C_3$  satisfies the conditions of Lemma 10 and, therefore,

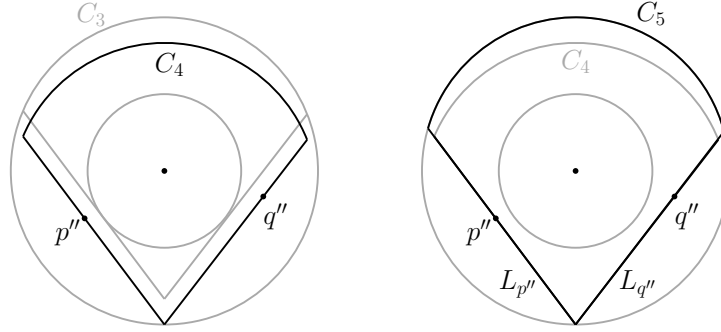
$$Dil(C) \leq \frac{|p'q'|_{C_3}}{|p'q'|} \leq Dil(C_3) \leq f(R/r).$$

Now assume that  $C_3$  is of type 2. We may assume without loss of generality that  $C_3$  is symmetric with respect to the  $y$ -axis, and the intersection point of  $L_{p'}$  and  $L_{q'}$  is on the negative  $y$ -axis. Translate  $C_3$  in the negative  $y$ -direction until it touches the outer circle. Denote the resulting translate by  $C_4$ . Let  $p''$  and  $q''$  be the two points on  $C_4$  that correspond to  $p'$  and  $q'$ , respectively. Then

$$Dil(C) \leq \frac{|p'q'|_{C_3}}{|p'q'|} = \frac{|p''q''|_{C_4}}{|p''q''|}.$$

We consider two cases.

**Case 1:** The lowest point of  $C_4$  is on the outer circle of  $Ann_{r,R}$ ; see the left figure below.



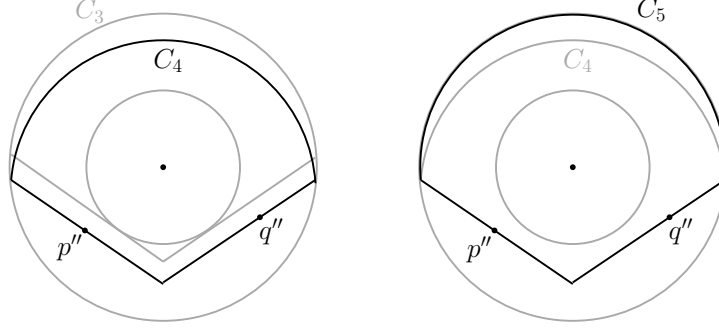
Let  $L_{p''}$  and  $L_{q''}$  be supporting lines of  $C_4$  through  $p''$  and  $q''$ , respectively, and let  $C_5$  be the convex cycle of maximum length in  $Ann_{r,R}$  that is between  $L_{p''}$  and  $L_{q''}$ ; see the right figure above. Observe that

$$Dil(C) \leq \frac{|p''q''|_{C_4}}{|p''q''|} \leq \frac{|p''q''|_{C_5}}{|p''q''|} \leq Dil(C_5).$$

Enlarge the inner circle of  $Ann_{r,R}$  such that it touches the two line segments of  $C_5$ . Denoting the radius of this enlarged circle by  $r'$ , it follows from Lemmas 10 and 4 that

$$Dil(C) \leq Dil(C_5) \leq f(R/r') \leq f(R/r).$$

**Case 2:** The leftmost and rightmost points of  $C_4$  are on the outer circle of  $Ann_{r,R}$ ; see the left figure below.



Let  $C_5$  be the convex cycle consisting of the two line segments of  $C_4$  and the upper arc on the outer circle connecting them; see the right figure above. Then

$$Dil(C) \leq \frac{|p''q''|_{C_4}}{|p''q''|} \leq \frac{|p''q''|_{C_5}}{|p''q''|} \leq Dil(C_5).$$

Enlarge the inner circle of  $Ann_{r,R}$  such that it touches the two line segments of  $C_5$ . Let  $r'$  be the radius of this enlarged circle. Since  $C_5$  satisfies the conditions of Lemma 9 for  $Ann_{r',R}$ , we have

$$Dil(C_5) \leq f(R/r') \leq f(R/r).$$

Thus, we have shown that  $Dil(C) \leq f(R/r)$ .

Until now we have assumed that neither  $p$  nor  $q$  is on the outer circle of  $Ann_{r,R}$ . Assume now that  $p$  or  $q$  is on this outer circle. Let  $\varepsilon > 0$  be an arbitrary real number and consider the annulus  $Ann_{r,R+\varepsilon}$ . Since neither  $p$  nor  $q$  is on the outer circle of this enlarged annulus, the analysis given above implies that

$$Dil(C) \leq f((R+\varepsilon)/r).$$

Thus, since this holds for any  $\varepsilon > 0$ , we have

$$Dil(C) \leq \inf_{\varepsilon > 0} f((R+\varepsilon)/r).$$

Since the function  $f$  is continuous, it follows from Lemma 4 that

$$Dil(C) \leq \inf_{\varepsilon > 0} f((R+\varepsilon)/r) = f(R/r).$$

This concludes the proof. ■

## 4 Angle-Constrained Convex Polyhedra in a Spherical Shell

Let  $r$  and  $R$  be real numbers with  $R > r > 0$ . Define  $Shell_{r,R}$  to be the *spherical shell* consisting of all points in  $\mathbb{R}^3$  that are on or between the two spheres of radii  $r$  and  $R$  that are centered at the origin. In other words,

$$Shell_{r,R} = \{(x, y, z) \in \mathbb{R}^3 : r \leq x^2 + y^2 + z^2 \leq R^2\}.$$

In this section, we consider convex simplicial polyhedra that contain the origin in their interiors and whose boundaries are contained in  $Shell_{r,R}$ . From Section 2.4, the skeletons of such polyhedra can have unbounded stretch factors.

Let  $\theta$  be a real number with  $0 < \theta < \pi/3$ . We say that a convex polyhedron  $P$  is  $\theta$ -angle-constrained, if the angles in all faces of  $P$  are at least  $\theta$ .

Let  $P$  be a convex simplicial polyhedron that contains the origin in its interior, whose boundary is contained in  $Shell_{r,R}$ , and that is  $\theta$ -angle-constrained. In this section, we prove that the stretch factor of the skeleton of  $P$  is bounded from above by a function of  $R/r$  and  $\theta$ . Our proof will use an improvement of a result by Karavelas and Guibas [7] about chains of triangles in  $\mathbb{R}^2$ ; see Lemma 12. We start by reviewing such chains.

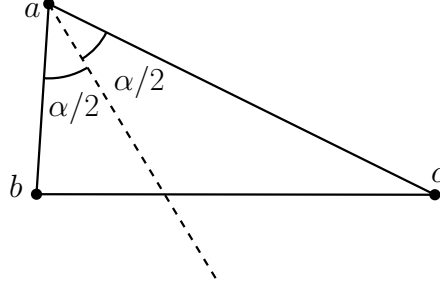
## 4.1 Chains of Triangles

Before we define chains of triangles, we prove a geometric lemma that will be used later in this section.

**Lemma 11** *Let  $a$ ,  $b$ , and  $c$  be three pairwise distinct points in the plane, and let  $\alpha = \angle(bac)$ . Then*

$$|ab| + |ac| \leq \frac{|bc|}{\sin(\alpha/2)}.$$

**Proof.** Consider the interior angle bisector of  $a$ ; see the figure below.



Let  $\ell_b$  be the distance between  $b$  and this bisector, and let  $\ell_c$  be the distance between  $c$  and this bisector. Then

$$|ab| + |ac| = \frac{\ell_b}{\sin(\alpha/2)} + \frac{\ell_c}{\sin(\alpha/2)} \leq \frac{|bc|}{\sin(\alpha/2)}.$$

■

Let  $p$  and  $q$  be two distinct points in  $\mathbb{R}^2$ , let  $k \geq 2$  be an integer, and consider a sequence  $\mathcal{T} = (T_1, T_2, \dots, T_k)$  of triangles in  $\mathbb{R}^2$ . The sequence  $\mathcal{T}$  is called a *chain of triangles with respect to  $p$  and  $q$* , if

1.  $p$  is a vertex of  $T_1$ , but not of  $T_2$ ,

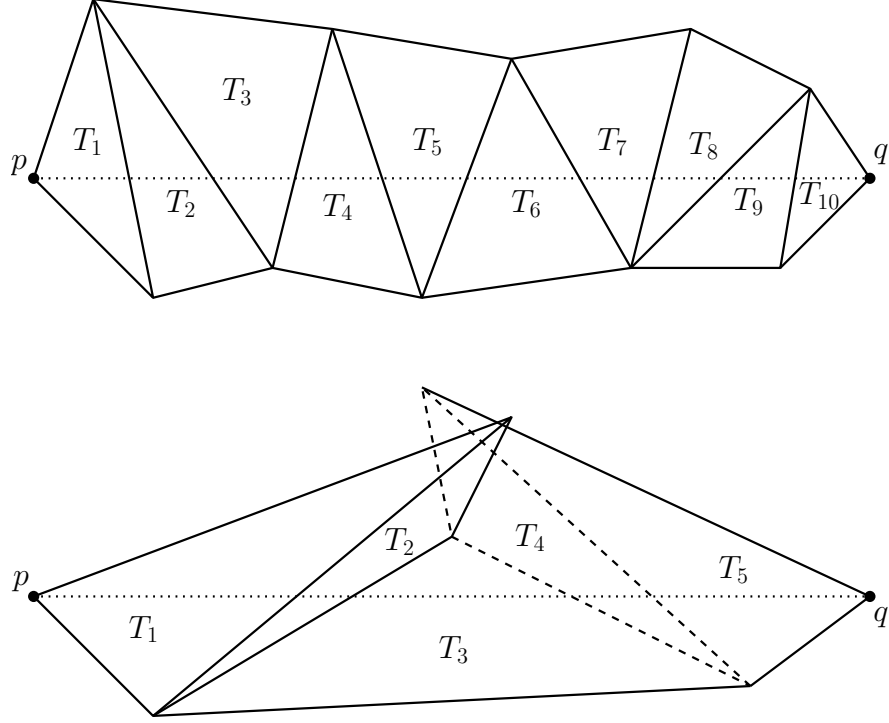


Figure 4: Two examples of chains of triangles with respect to the points  $p$  and  $q$ . For clarity, the triangle  $T_4$  in the second example is dashed.

- 
2.  $q$  is a vertex of  $T_k$ , but not of  $T_{k-1}$ ,
  3. for each  $i$  with  $1 \leq i < k$ , the interiors of the triangles  $T_i$  and  $T_{i+1}$  are disjoint and these triangles share an edge, and
  4. for each  $i$  with  $1 \leq i \leq k$ , the line segment  $pq$  intersects the interior of  $T_i$ .

See Figure 4 for examples.

Let  $G(\mathcal{T})$  be the graph whose vertex and edge sets consist of all vertices and edges of the  $k$  triangles in  $\mathcal{T}$ , respectively. The length of each edge in this graph is equal to the Euclidean distance between its vertices. The length of a shortest path in  $G(\mathcal{T})$  is denoted by  $|pq|_{G(\mathcal{T})}$ .

**Lemma 12** *Let  $\theta$  be a real number with  $0 < \theta < \pi/3$ , let  $p$  and  $q$  be two distinct points in the plane, and let  $\mathcal{T}$  be a chain of triangles with respect to  $p$  and  $q$ . Assume that all angles in any of the triangles in  $\mathcal{T}$  are at least  $\theta$ . Then*

$$|pq|_{G(\mathcal{T})} \leq \frac{1 + 1/\sin(\theta/2)}{2} \cdot |pq|.$$

**Proof.** We assume, without loss of generality, that the line segment  $pq$  is on the  $x$ -axis and  $p$  is to the left of  $q$ . We start by constructing a preliminary path in  $G(\mathcal{T})$  from  $p$  to  $q$  (this is the same path as in Karavelas and Guibas [7]):



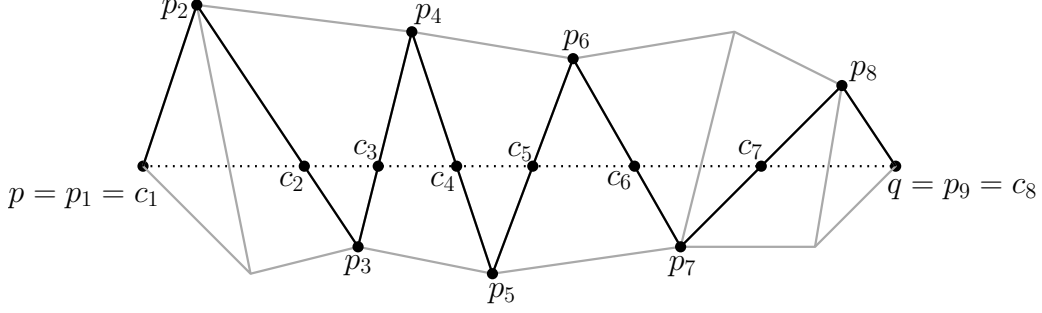


Figure 5: The path  $\Pi = (p = p_1, p_2, \dots, p_9 = q)$  in the first chain of triangles in Figure 4. The segments  $c_1c_2$ ,  $c_6c_7$ , and  $c_7c_8$  belong to group 1, the segments  $c_3c_4$  and  $c_5c_6$  belong to group 2, and the segments  $c_2c_3$  and  $c_4c_5$  belong to group 3. The path  $\Pi'$  is equal to  $(p = p_1, p_2, p_3, p_5, p_7, p_8, p_9 = q)$ .

1. Let  $pr$  be one of the two edges of the triangle  $T_1$  with endpoint  $p$ . We initialize the path to be  $(p, r)$ .
2. Consider the current path and let  $r$  be its last point. Assume that  $r \neq q$ .
  - (a) If  $r$  is below the  $x$ -axis, then consider all edges in  $G(\mathcal{T})$  that have  $r$  as an endpoint and whose other endpoint is on or above the  $x$ -axis. Let  $rr'$  be the “rightmost” of these edges, i.e., the edge among these whose angle with the positive  $x$ -axis is minimum. Then we extend the path by the edge  $rr'$ , i.e., we add the point  $r'$  at the end of the current path.
  - (b) If  $r$  is above the  $x$ -axis, then consider all edges in  $G(\mathcal{T})$  that have  $r$  as an endpoint and whose other endpoint is on or below the  $x$ -axis. Let  $rr'$  be the “rightmost” of these edges, i.e., the edge among these whose angle with the positive  $x$ -axis is maximum. Then we extend the path by the edge  $rr'$ , i.e., we add the point  $r'$  at the end of the current path.

Number the triangles in  $\mathcal{T}$  as  $T_1, T_2, \dots, T_k$ , in the order in which they are intersected by the line segment from  $p$  to  $q$ . Then the point  $r'$  is a vertex of a triangle in  $\mathcal{T}$  that has a larger index than the index of any triangle that contains the vertex  $r$ . Therefore, if we continue extending the path, it will reach the point  $q$ . Denote the resulting path by  $\Pi = (p = p_1, p_2, \dots, p_\ell = q)$ ; see Figure 5.

As a warming-up, we prove an upper bound on the length of the path  $\Pi$ . For each  $i$  with  $1 \leq i < \ell$ , let  $c_i$  be the intersection between the line segments  $pq$  and  $p_i p_{i+1}$ . Then  $|pq|_{G(\mathcal{T})}$  is at most the length of the path  $\Pi$ , i.e.,

$$|pq|_{G(\mathcal{T})} \leq \sum_{i=1}^{\ell-1} |p_i p_{i+1}| = \sum_{i=1}^{\ell-1} (|c_i p_{i+1}| + |p_{i+1} c_{i+1}|).$$

Let  $\alpha_i = \angle(c_i p_{i+1} c_{i+1})$ . Since  $\alpha_i \geq \theta$ , it follows from Lemma 11 that

$$|c_i p_{i+1}| + |p_{i+1} c_{i+1}| \leq \frac{|c_i c_{i+1}|}{\sin(\alpha_i/2)} \leq \frac{|c_i c_{i+1}|}{\sin(\theta/2)}.$$

Therefore,

$$|pq|_{G(\mathcal{T})} \leq \sum_{i=1}^{\ell-1} \frac{|c_i c_{i+1}|}{\sin(\theta/2)} = \frac{|pq|}{\sin(\theta/2)}.$$

To improve the upper bound on  $|pq|_{G(\mathcal{T})}$ , we divide the line segments  $c_i c_{i+1}$ ,  $1 \leq i < \ell$ , into three groups: A segment  $c_i c_{i+1}$  belongs to *group 1* if its relative interior intersects an edge of some triangle of the chain  $\mathcal{T}$ . If the relative interior of  $c_i c_{i+1}$  is entirely contained in one of the triangles of  $\mathcal{T}$  and the point  $p_{i+1}$  is on or above the  $x$ -axis, then  $c_i c_{i+1}$  belongs to *group 2*. Otherwise,  $c_i c_{i+1}$  belongs to *group 3*. Refer to Figure 5 for an illustration. For  $j = 1, 2, 3$ , let  $X_j$  denote the total length of all line segments  $c_i c_{i+1}$  in group  $j$ . We may assume without loss of generality that  $X_3 \leq X_2$ .

Consider again the path  $\Pi = (p = p_1, p_2, \dots, p_\ell = q)$ . For each  $i$  such that  $c_i c_{i+1}$  belongs to group 2, we replace the subpath  $(p_i, p_{i+1}, p_{i+2})$  in  $\Pi$  by the short-cut  $p_i p_{i+2}$ ; refer to Figure 5. Let  $\Pi'$  denote the resulting path from  $p$  to  $q$ .

For each line segment  $c_i c_{i+1}$  in group 1, we have  $\alpha_i \geq 2\theta$ . If  $c_i c_{i+1}$  is in group 2, then

$$|p_i p_{i+2}| \leq |p_i c_i| + |c_i c_{i+1}| + |c_{i+1} p_{i+2}|.$$

It follows that

$$|pq|_{G(\mathcal{T})} \leq |\Pi'| \leq \frac{X_1}{\sin \theta} + X_2 + \frac{X_3}{\sin(\theta/2)}.$$

Recall that  $0 \leq X_3 \leq X_2$ . This inequality is equivalent to

$$\begin{aligned} X_2 + \frac{X_3}{\sin(\theta/2)} &\leq \frac{1}{2} \left( 1 + \frac{1}{\sin(\theta/2)} \right) (X_2 + X_3) \\ &= \frac{1}{2} \left( 1 + \frac{1}{\sin(\theta/2)} \right) (|pq| - X_1) \end{aligned}$$

implying that

$$|pq|_{G(\mathcal{T})} \leq \left( \frac{1}{\sin \theta} - \frac{1}{2} - \frac{1}{2 \sin(\theta/2)} \right) X_1 + \frac{1}{2} \left( 1 + \frac{1}{\sin(\theta/2)} \right) |pq|. \quad (3)$$

The function  $g(\theta) = 1/\sin \theta - 1/2 - 1/(2 \sin(\theta/2))$  is negative for  $0 < \theta < \pi/3$ . To prove this, using  $\sin \theta = 2 \sin(\theta/2) \cos(\theta/2)$  and a straightforward calculation, we observe that  $g(\theta) < 0$  if and only if

$$\frac{\sin \theta}{2} + \cos(\theta/2) > 1.$$

The left-hand side in the above inequality has a positive derivative for  $0 < \theta < \pi/3$  (this can be verified using  $\cos \theta = 1 - 2 \sin^2(\theta/2)$ ); thus,

$$\frac{\sin \theta}{2} + \cos(\theta/2) > \frac{\sin 0}{2} + \cos(0/2) = 1.$$

We conclude that, since  $g(\theta) < 0$ , the first term on the right-hand side in (3) is non-positive, implying that

$$|pq|_{G(\mathcal{T})} \leq \frac{1 + 1/\sin(\theta/2)}{2} \cdot |pq|.$$

This completes the proof. ■

## 4.2 Angle-Constrained Convex Polyhedra

Let  $\theta$  be a real number with  $0 < \theta < \pi/3$  and let  $P$  be a convex simplicial polyhedron that is  $\theta$ -angle-constrained. In this section, we bound the ratio of the shortest-path distance  $|pq|_{\text{skel}(P)}$  between  $p$  and  $q$  in the skeleton of  $P$  and the shortest-path distance  $|pq|_{\partial P}$  between  $p$  and  $q$  along the surface of  $P$ .

Let  $p$  and  $q$  be two distinct vertices of  $P$  and consider the shortest path  $\Pi_{pq}$  along the surface of  $P$  from  $p$  to  $q$ . Except for  $p$  and  $q$ , this path does not contain any vertex of  $P$ ; see Sharir and Schorr [9]. Let  $T_1, T_2, \dots, T_k$  be the sequence of faces of  $P$  that this path passes through. Let  $\mathcal{T}' = (T'_1, T'_2, \dots, T'_k)$  be the sequence of triangles obtained from an edge-unfolding of the triangles  $T_1, T_2, \dots, T_k$ . Let  $p'$  and  $q'$  be the vertices of  $T'_1$  and  $T'_k$  corresponding to  $p$  and  $q$ , respectively. Sharir and Schorr [9] (see also Agarwal *et al.* [1]) have shown that

- $\mathcal{T}'$  is a chain of triangles with respect to  $p'$  and  $q'$ , as defined in Section 4.1, and
- the path  $\Pi_{pq}$  along  $\partial P$  unfolds to the line segment  $p'q'$ , i.e.,  $|pq|_{\partial P} = |\Pi_{pq}| = |p'q'|$ .

Consider the graph  $G(\mathcal{T}')$  that is defined by the chain  $\mathcal{T}'$ ; see Section 4.1. Observe that  $|pq|_{\text{skel}(P)}$  is at most the shortest-path distance between  $p$  and  $q$  in the graph consisting of all vertices and edges of the triangles  $T_1, T_2, \dots, T_k$ . The latter shortest-path distance is equal to  $|p'q'|_{G(\mathcal{T}')}$ . Thus, using Lemma 12, we obtain

$$\begin{aligned} |pq|_{\text{skel}(P)} &\leq |p'q'|_{G(\mathcal{T}')} \\ &\leq \frac{1 + 1/\sin(\theta/2)}{2} \cdot |p'q'| \\ &= \frac{1 + 1/\sin(\theta/2)}{2} \cdot |pq|_{\partial P}. \end{aligned}$$

We have proved the following result:

**Lemma 13** *Let  $\theta$  be a real number with  $0 < \theta < \pi/3$  and let  $P$  be a  $\theta$ -angle-constrained convex simplicial polyhedron. For any two distinct vertices  $p$  and  $q$  of  $P$ , we have*

$$|pq|_{\text{skel}(P)} \leq \frac{1 + 1/\sin(\theta/2)}{2} \cdot |pq|_{\partial P}.$$

### 4.3 Angle-Constrained Convex Polyhedra in a Spherical Shell

We are now ready to prove the main result of Section 4:

**Theorem 5** *Let  $r$ ,  $R$ , and  $\theta$  be real numbers with  $R > r > 0$  and  $0 < \theta < \pi/3$ , and let  $P$  be a  $\theta$ -angle-constrained convex simplicial polyhedron that contains the origin and whose boundary is contained in the spherical shell  $\text{Shell}_{r,R}$ . Then the skeleton of  $P$  is a  $t$ -spanner of the vertex set of  $P$ , where*

$$t = \frac{1 + 1/\sin(\theta/2)}{2} \left( \sqrt{(R/r)^2 - 1} + (R/r) \arcsin(r/R) \right).$$

**Proof.** Let  $p$  and  $q$  be two distinct vertices of  $P$ . By Lemma 13, we have

$$|pq|_{\text{ske}(P)} \leq \frac{1 + 1/\sin(\theta/2)}{2} \cdot |pq|_{\partial P}.$$

Let  $H_{pq}$  be the plane through  $p$ ,  $q$ , and the origin, and let  $Q_{pq}$  be the intersection of  $P$  and  $H_{pq}$ . Then

$$|pq|_{\partial P} \leq |pq|_{\partial Q_{pq}}.$$

Since  $Q_{pq}$  is a convex polygon satisfying the conditions of Theorem 3, we have

$$|pq|_{\partial Q_{pq}} \leq \left( \sqrt{(R/r)^2 - 1} + (R/r) \arcsin(r/R) \right) \cdot |pq|.$$

■

## 5 Concluding Remarks

We have considered the problem of bounding the stretch factor of the skeleton of a convex simplicial polyhedron  $P$  in  $\mathbb{R}^3$ . If the vertices of  $P$  are on a sphere, then this stretch factor is at most  $0.999 \cdot \pi$ , which is  $\pi/2$  times the currently best known upper bound on the stretch factor of the Delaunay triangulation in  $\mathbb{R}^2$ . We obtained this result from Xia's upper bound on the stretch factor of chains of disks in [11]. Observe that Xia's result implies an upper bound on the stretch factor of the Delaunay triangulation. The converse, however, may not be true, because the chains of disks that arise in the analysis of the Delaunay triangulation are much more restricted than general chains of disks; see for example Figure 2 in [11]. Thus, an improved upper bound on the stretch factor of the Delaunay triangulation may not imply an improved upper bound on the stretch factor of the skeleton of  $P$ . Nevertheless, we make the following conjecture: Let  $t^*$  be a real number such that the stretch factor of any Delaunay triangulation in  $\mathbb{R}^2$  is at most  $t^*$ . Then the stretch factor of the skeleton of any convex polyhedron in  $\mathbb{R}^3$ , all of whose vertices are on a sphere, is at most  $t^* \cdot \pi/2$ .

We have shown that the skeleton of a convex simplicial polyhedron  $P$  whose vertices are “almost” on a sphere may have an unbounded stretch factor. For the case when  $P$  contains

the origin, its boundary is contained in the spherical shell  $Shell_{r,R}$ , and the angles in all faces are at least  $\theta$ , we have shown that the stretch factor of  $P$ 's skeleton is bounded from above by a function that depends only on  $R/r$  and  $\theta$ . We leave as an open problem to find other classes of convex polyhedra whose skeletons have bounded stretch factor.

## Acknowledgments

Part of this work was done at the *Third Annual Workshop on Geometry and Graphs*, held at the Bellairs Research Institute in Barbados, March 8–13, 2015. We thank the other workshop participants for their helpful comments.

We thank the anonymous referees for their useful comments. We especially thank one of the referees for simplifying the proofs of Lemmas 3 and 11, and for suggesting the use of short-cuts in the proof of Lemma 12.

## References

- [1] P. K. Agarwal, B. Aronov, J. O'Rourke, and C. A. Schevon. Star unfolding of a polytope with applications. *SIAM Journal on Computing*, 26:1689–1713, 1997.
- [2] P. Bose, S. Pratt, and M. Smid. The convex hull of points on a sphere is a spanner. In *Proceedings of the 26th Canadian Conference on Computational Geometry*, pages 244–250, 2014.
- [3] K. Q. Brown. *Geometric Transforms for Fast Geometric Algorithms*. Ph.D. thesis, Carnegie-Mellon University, 1979.
- [4] D. P. Dobkin, S. J. Friedman, and K. J. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete & Computational Geometry*, 5:399–407, 1990.
- [5] A. Ebbels-Baumann, A. Grüne, and R. Klein. Geometric dilation of closed planar curves: New lower bounds. *Computational Geometry: Theory and Applications*, 37:188–208, 2007.
- [6] A. Grüne. *Geometric Dilation and Halving Distance*. Ph.D. thesis, Universität Bonn, Germany, 2006.
- [7] M. I. Karavelas and L. J. Guibas. Static and kinetic geometric spanners with applications. In *Proceedings of the 12th ACM-SIAM Symposium on Discrete Algorithms*, pages 168–176, 2001.
- [8] G. Narasimhan and M. Smid. *Geometric Spanner Networks*. Cambridge University Press, Cambridge, UK, 2007.

- [9] M. Sharir and A. Schorr. On shortest paths in polyhedral spaces. *SIAM Journal on Computing*, 15:193–215, 1986.
- [10] V. A. Toponogov. *Differential Geometry of Curves and Surfaces*. Birkhäuser, Boston, 2006.
- [11] G. Xia. The stretch factor of the Delaunay triangulation is less than 1.998. *SIAM Journal on Computing*, 42:1620–1659, 2013.